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NOTE ON POLYAGROUPS

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Abstract. In the paper the following proposition is proved. Let $k > 1$, $s > 1$, $n = k \cdot s + 1$ and let (Q, A) be an $n-$groupoid. Then, (Q, A) is a **polyagroup of the type** $(s, n-1)$ iff the following statements hold: (i) (Q, A) is an $< i, s+i >$ $-$associative $n-$groupoid for all $i \in \{1, \dots, s\}$; $< 1, n >$ $-$associative $n-$groupoid; (iii) for every $a_1^n \in Q$ there is **at least one** $x \in Q$ and **at least one** $y \in Q$ such that the following equalities hold $A(x, a_1^{n-1}) = a_n$ and $A(a_1^{n-1}, y) = a_n$; and (iv) for every $a_1^n \in Q$ and for all $i \in \{2, \dots, s\} \cup \{(k-1) \cdot s+2, \dots, k \cdot s\}$ there is **exactly one** $x_i \in Q$ such that the following equality holds $A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n$. *[* The case $s = 1$ (: $(i) - (iii)$) is discribed in [4]. *]*

1. Preliminaries

1.1. Definitions: *Let $k > 1$, $s \geq 1$, $n = k \cdot s + 1$ and let (Q, A) be an n-groupoid. Then: a) We say that (Q, A) is an* s-**associative** n-**groupoid** *iff it is $< u, v >$-associative for all $u, v \in \{1, \dots, n\}$ such that $u \equiv v (mod\ s)$ (cf. [1,2]); b) We say that (Q, A) is an* i-**partially** s-**associative** *(briefly: iPs-associative)* n-**groupoid**, *$i \in \{1, \dots, s\}$, iff it is $< i, t \cdot s + i >$-associative for all $t \in \{1, \dots, k\}$ such that $t \cdot s + i \leq k \cdot s + 1$ c) We say that (Q, A) is a* **polyagroup of the type** $(s, n-1)$ *iff is an s-associative n-groupoid and an n-quasigroup (cf. [1,2]); and d) We say that (Q, A) is an* **near$-P$-polyagroup** *(briefly: NP-polyagroup)***of the type** $(s, n-1)$ *iff is an Ps-associative n-groupoid and for every $j \in \{t \cdot s + 1 \mid t \in \{0, 1, \dots, k\}\}$ and for all $a_1^n \in Q$ there is exactly one $x_j \in Q$ such that the equality*

$$A(a_1^{j-1}, x_j, a_j^{n-1}) = a_n$$

holds (cf. [6]).

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$1Ps$-associative (Ps-associative) n-groupoid introduced in [6].

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By 1.1, we conclude that the following proposition holds:

1.2. Proposition: *Let $k > 1,\ s \geq 1,\ n = k \cdot s + 1$ and let (Q, A) n-groupoid. Then: (Q, A) is an polyagroup of the type $(s, n-1)$ iff is an iPs-associative n-groupoid for all $i \in \{1, \ldots, s\}$ and is an n-quasigroup.*

Remark: *Every polyagroup of the type $(s, n-1)$ is an NP-polyagroup of the type $(s, n-1)$.*

2. Auxiliary propositions

2.1. Proposition [6]: *Let $k > 1,\ s \geq 1,\ n = k \cdot s + 1$ and let (Q, A) be an n-groupoid. Then, (Q, A), is a* **near-P-polyagroup** *(briefly: NP-polyagroup)* **of the type** $(s, n-1)$ *iff the following statements hold:*

(*i*) *(Q, A) is an $< 1, s+1 >$-associative n-groupoid;*

(*ii*) *(Q, A) is an $< 1, n >$-associative n-groupoid; and*

(*iii*) *For every $a_1^n \in Q$ there is* **at least one** *$x \in Q$ and* **at least one** *$y \in Q$ such that the following equalities hold*

$$A(x, a_1^{n-1}) = a_n \text{ and } A(a_1^{n-1}, y) = a_n \ \Box$$

Remark: *For $s = 1$ Proposition 2.1 is proved in [4].*

2.2. Proposition: *Let $k > 1,\ s > 1,\ n = k \cdot s + 1,\ i \in \{1, \ldots, s\}$ and let (Q, A) be an n-groupoid. Also, let*

(a) the $< i, s+i >$-associative law holds in the (Q, A); and

(b) for every $x, y, a_1^{n-1} \in Q$ the following implication holds

$$A(a_1^{i-1}, x, a_i^{n-1}) = A(a_1^{i-1}, y, a_i^{n-1}) \Rightarrow x = y$$

Then (Q, A) is an iPs-associative n-groupoid.

Remark: For $k = 2$ and $i \in \{2, \ldots, s\}$, (Q, A) is an iPs-associative n-groupoid iff (a).

Sketch of the proof.

$$A(a_1^{i-1}, A(a_i^{n+i-1}), a_{n+i}^{2n-1}) = A(a_1^{i-1}, a_i^{s+i-1}, A(a_{s+i}^{n+s+i-1}), a_{n+s+i}^{2n-1}) \Rightarrow$$

$$A(b_{s+1}^{s+i-1}, b_1^s, A(a_1^{i-1}, A(a_i^{n+i-1}), a_{n+i}^{2n-s-1}, a_{2n-s}^{2n-1}), b_{s+i}^{n-1}) =$$

$$A(b_{s+1}^{s+i-1}, b_1^s, A(a_1^{i-1}, a_i^{s+i-1}, A(a_{s+i}^{n+s+i-1}), a_{n+s+i}^{2n-s-1}, a_{2n-s}^{2n-1}), b_{s+i}^{n-1}) \Rightarrow$$

$$A(b_{s+1}^{s+i-1}, A(b_1^s, a_1^{i-1}, A(a_1^{n+i-1}), a_{n+i}^{2n-s-1}), a_{2n-s}^{2n-1}, b_{s+i}^{n-1}) =$$

$$A(b_{s+1}^{s+i-1}, A(b_1^s, a_1^{i-1}, a_i^{s+i-1}, A(a_{s+i}^{n+s+i-1}), a_{n+s+i}^{2n-s-1}), a_{2n-s}^{2n-1}, b_{s+i}^{n-1}) \Rightarrow$$

$$A(b_1^s, a_1^{i-1}, A(a_i^{n+i-1}), a_{n+i}^{2n-s-1}) = A(b_1^s, a_1^{i-1}, a_i^{s+i-1}, A(a_{s+i}^{n+s+i-1}), a_{n+s+i}^{2n-s-1}).$$

(See, also [3,6].)

2.3. Proposition: *Let $k > 2,\ s > 1,\ n = k \cdot s + 1,\ i \in \{1, \ldots, s\}$ and let (Q, A) be an n-groupoid. Also, let*

(1) (Q, A) *is an* iPs*-associative* $(i \in \{2, \ldots, s\})$ n*-groupoid;*

(2) For every $x, y, a_1^{n-1} \in Q$ *the following implication holds* $A(a_1^{i-1}, x, a_i^{n-1}) = A(a_1^{i-1}, y, a_i^{n-1}) \Rightarrow x = y$; *and*

(3) For every $x, y, a_1^{n-1} \in Q$ *the following implication holds*

$$A(a_1^{(k-1)\cdot s+i-1}, x, a_{(k-1)\cdot s+i)}^{k\cdot s}) = A(a_1^{(k-1)\cdot s+i-1}, y, a_{(k-1)\cdot s+i)}^{k\cdot s}) \Rightarrow x = y.$$

Then, for every $x, y, a_1^{n-1} \in Q$ *and for all* $t \in \{1, \ldots, k-2\}$ *the following implication holds*

$$A(a_1^{t\cdot s+i-1}, x, a_{t\cdot s+i)}^{k\cdot s}) = A(a_1^{t\cdot s+i-1}, y, a_{t\cdot s+i)}^{k\cdot s}) \Rightarrow x = y.$$

Remark: $\Delta = ((k-1)\cdot s + i) - i = (k-1)\cdot s$. *For* $k = 2$, $\Delta = s$.

Sketch of the proof.

$$A(a_1^{t\cdot s+i-1}, x, b_1^{(k-t)\cdot s-i+1}) = A(a_1^{t\cdot s+i-1}, y, b_1^{(k-t)\cdot s-i+1}) \Rightarrow$$

$$A(c_1^{i-1}, d_1^{(k-t-1)\cdot s}, A(a_1^{t\cdot s+i-1}, x, b_1^{(k-t)\cdot s-i+1}), c_i^{t\cdot s+i-1}, d_{(k-t-1)\cdot s+1}^{(k-t)\cdot s-i+1}) =$$

$$A(c_1^{i-1}, d_1^{(k-t-1)\cdot s}, A(a_1^{t\cdot s+i-1}, y, b_1^{(k-t)\cdot s-i+1}), c_i^{t\cdot s+i-1}, d_{(k-t-1)\cdot s+1}^{(k-t)\cdot s-i+1}) \overset{(1)}{\Rightarrow}$$

$$A(c_1^{i-1}, A(d_1^{(k-t-1)\cdot s}, a_1^{t\cdot s+i-1}, x, b_1^{s-i+1}), b_{s-i+2}^{(k-t)\cdot s-i+1}, c_i^{t\cdot s+i-1}, d_{(k-t-1)\cdot s+1}^{(k-t)\cdot s-i+1}) =$$

$$A(c_1^{i-1}, A(d_1^{(k-t-1)\cdot s}, a_1^{t\cdot s+i-1}, y, b_1^{s-i+1}), b_{s-i+2}^{(k-t)\cdot s-i+1}, c_i^{t\cdot s+i-1}, d_{(k-t-1)\cdot s+1}^{(k-t)\cdot s-i+1}) \overset{(2)}{\Rightarrow}$$

$$A(d_1^{(k-t-1)\cdot s}, a_1^{t\cdot s+i-1}, x, b_1^{s-i+1}) = A(d_1^{(k-t-1)\cdot s}, a_1^{t\cdot s+i-1}, y, b_1^{s-i+1}) \overset{(3)}{\Rightarrow}$$

$x = y$. □

2.4. Proposition: *Let* $k > 2$, $s > 1$, $n = k\cdot s + 1$ *and let* (Q, A) *be an* n*-groupoid. Also, let*

($\overline{1}$*)* (Q, A) *is an* iPs*-associative* $(i \in \{2, \ldots, s\})$ n*-groupoid;*

($\overline{2}$*) For every* $a_1^n \in Q$ *there is exactly one* $x \in Q$ *such that the following equality holds*

$$A(a_1^{i-1}, x, a_i^{n-1}) = a_n;$$

and

($\overline{3}$*) For every* $a_1^{k\cdot s+1} \in Q$ *there is exactly one* $y \in Q$ *such that the following equality holds*

$$A(a_1^{(k-1)\cdot s+i-1}, y, a_{(k-1)\cdot s+i}^{k\cdot s}) = a_{k\cdot s+1}.$$

Then, for every $a_1^{k\cdot s+1} \in Q$ *and for all* $t \in \{1, \ldots, k-2\}$ *there is* **at least one** $z \in Q$ *such that the following equality holds*

$$A(a_1^{t\cdot s+i-1}, z, a_{t\cdot s+i}^{k\cdot s}) = a_{k\cdot s+1}.$$

Sketch of the proof.

$A(a_1^{t\cdot s+i-1}, z, b_1^{(k-t)\cdot s-i+1}) = b_{(k-t)\cdot s-i+2} \overset{2\cdot 3}{\Leftrightarrow}$

$A(c_1^{i-1}, d_1^{(k-t-1)\cdot s}, A(a_1^{t\cdot s+i-1}, z, b_1^{(k-t)\cdot s-i+1}), c_i^{t\cdot s+i-1}, d_{(k-t-1)\cdot s+1}^{(k-t)\cdot s-i+1}) =$

$A(c_1^{i-1}, d_1^{(k-t-1)\cdot s}, b_{(k-t)\cdot s-i+2}, c_i^{t\cdot s+i-1}, d_{(k-t-1)\cdot s+1}^{(k-t)\cdot s-i+1}) \overset{(1)}{\Leftrightarrow}$

$A(c_1^{i-1}, A(d_1^{(k-t-1)\cdot s}, a_1^{t\cdot s+i-1}, z, b_1^{s-i+1}), b_{s-i+2}^{(k-t)\cdot s-i+1}, c_i^{t\cdot s+i-1}, d_{(k-t-1)\cdot s+1}^{(k-t)\cdot s-i+1}) =$

$A(c_1^{i-1}, d_1^{(k-t-1)\cdot s}, b_{(k-t)\cdot s-i+2}, c_i^{t\cdot s+i-1}, d_{(k-t-1)\cdot s+1}^{(k-t)\cdot s-i+1})$. $\square$

3. Result

3.1. Theorem: *Let $k > 1, s > 1,\ n = k\cdot s+1$ and let (Q, A) be an n-groupoid. Then, (Q, A) is a* **polyagroup of the type** $(s, n-1)$ *iff the folllowing statements hold:*

(i) *(Q, A) is an $< i, s+i >$-associative n-groupoid for all $i \in \{1, \ldots, s\}$;*

(ii) *(Q, A) is an $< 1, n >$-associative n-groupoid;*

(iii) *For every $a_1^n \in Q$ there is* **at least one** *$x \in Q$ and* **at least one** *$y \in Q$ such that the following equalities hold*
$$A(x, a_1^{n-1}) = a_n \text{ and } A(a_1^{n-1}, y) = a_n; \text{ and}$$

(iv) *For every $a_1^n \in Q$ and for all $j \in \{2, \ldots, s\} \cup \{(k-1)\cdot s+2, \ldots, k\cdot s\}$ there is* **exactly one** *$x_j \in Q$ such that the following equality holds*
$$A(a_1^{j-1}, x_j, a_j^{n-1}) = a_n.$$

Proof. 1)$\Rightarrow$: Let (Q, A) be a polyagroup of the type $(s, n-1)$ and $s > 1$. Then, by the Definition 1.1, immediately we conclude that the statements $(i)-(iv)$ hold.

2)$\Leftarrow$: Firstly we prove that under assumtions the following statements hold:

1° (Q, A) is an near$-P$-polyagroup;

2° (Q, A) is an iPs-associative n-groupoid for all $i \in \{2, \ldots, s\}$; and

3° (Q, A) is an n-quasigroup.

The proof of the statement of 1° :

Bi (i) for $i = 1$, (ii), (iii) and Proposition 2.1.

The proof of the statement of 2° :

$a)$ $k = 2$: By (i).

$b)$ $k > 2$: By (i) for $i \in \{2, \ldots, s\}, (iv)$ and Proposition 2.2.

The proof of the statement of 3° :

$a)$ $k = 2$: By (iv).

$b)$ $k > 2$: By 1°, 2°, (iv), Proposition 2.3 and Proposition 2.4.

By $1^\circ - 3^\circ$ and Proposition 1.2, we conclude that the n-groupoid (Q, A) is a polyagroup of the type $(s, n-1)$. $\square$

3.2. Remark: *The case* $s = 1$ (: $(i) - (iii)$) *is described in [4]. See, also [5].*

4. References

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